2 Dynamics: An Introduction

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The word *dynamics* simply refers to the way a system changes or "behaves" as time passes. In the scientific literature, the use of this word may merely indicate that the author wishes to consider some system as evolving, rather than static. Or the author may refer to an attempt to formulate a more precise (either quantitative or qualitative) relation between an increasing time parameter and specific measurable elements of the system. Here, a large body of mathematics called *dynamical systems* becomes relevant. This chapter introduces the reader to certain basics of mathematical dynamical systems that will be useful in understanding the various modeling problems treated in the rest of this book.

We begin with a little background. For more details, the reader is referred to the survey article (Hirsch, 1984). Terms that appear in *italic* type, if not defined where they appear, are defined in the Glossary at the end of the book.

First, a *system* is some collection of related parts that we perceive as a single entity. For example, the following are familiar systems: the solar system, the capitalist system, the decimal system, the nervous system, the telephone system. Hirsch notes:

> A *dynamical* system is one which changes in time; what changes is the *state* of the system. The capitalist system is dynamical (according to Marx), while the decimal system is (we hope) not dynamical. A mathematical dynamical system consists of the space of [all possible] states of the system together with a rule called the *dynamic* for determining the state which corresponds at a given future time to a given present state. Determining such rules for various natural systems is a central problem of science. Once the dynamic is given, it is the task of mathematical dynamical systems theory to investigate the patterns of how states change in the long run. (Hirsch, 1984, p. 3).

Mathematical analysis requires that the state of a system be described by some clearly defined set of variables that may change as a function of time. A state is then identified with a choice of value for each of these variables. The collection of all possible (or relevant) values of these variables is called the *state space* (or sometimes *phase space*).

The most important dynamical system in scientific history is the solar system. The sun, planets, and moon are the parts of the system, the states are their possible configurations (and velocities), and the basic problem is to find the dynamic by which one can predict future events like eclipses. Historically this has been done by constructing various geometric or mathematical models for the system, e.g., those of Ptolemy, Copernicus, Brahe, Kepler.

After Galileo, Newton, and Leibnitz, the concepts of instant, velocity, and acceleration permitted the cosmos to be modeled by means of simple mathematical laws in the form of *differential equations*. From these, the visible behavior of the planets could be mathematically deduced with the help of the techniques of calculus. In the 18th and early 19th centuries, Euler, Laplace, Lagrange, the Bernoullis, and others developed "Newtonian" mechanics and the mathematics of differential equations (see section 2.1), used with great success to model an ever-increasing number of different physical systems.

The technique of formulating physical laws by means of differential equations (whose solutions then give the behavior of the system for all time) was so powerful that it was tempting to think of the entire universe as a giant mechanism ruled by a collection of differential equations based on a small number of simple laws. Since the solutions of a differential equation depend on the starting values assigned to the variables, it would then simply be a matter of specifying the *initial conditions*, e.g., the positions and velocities of all the particles in the universe, to then be able to predict with certainty all future behavior of every particle.

Today we know that *sensitivity to initial conditions* makes this impossible in principle, and, even for very small systems with only a few variables, there is another (related) serious difficulty inherent in this program: most differential equations cannot be solved exactly by means of mathematical formulas. For example, to this day the motion of three (or more) point masses in space acting under the influence of their mutual gravitational attraction is understood only in special cases, even though it is a simple matter to write down the differential equations governing such motion.

This profound difficulty remained unapproachable until in 1881 Henri Poincaré published the first of a series of papers inventing the point of view of what we now call dynamical systems theory: the *qualitative* study of differential equations. Rather than seeking a formula for each solution as a function of time, he proposed to study the collection of all solutions, thought of as curves or *trajectories* in state space, for all time and all initial conditions at once. This was a more geometric approach to the subject in that it appealed to intuitions about space, motion, and proximity to interpret these systems. This work also motivated his invention of a new discipline now called algebraic topology. Poincaré emphasized the importance of new themes from this point of view: *stability, periodic trajectories, recurrence,* and *generic behavior.*

One of the prime motivating questions was (and still is): Is the solar system stable? That is, will two of the planets ever collide, or will one ever escape from or fall into the sun? If we alter the mass of one of the planets or change its position slightly, will that lead to a drastic change in the trajectories? Or, can we be sure that, except for tidal friction and solar evolution, the solar system will continue as it is without catastrophe, even if small outside perturbations occur?

These are qualitative questions because we are not asking for specific values of position or velocity, but rather for general global features of the system over long time periods. This viewpoint requires thinking of the space of all possible states of the system as a geometric space in which the solution trajectories lie (as described below), and then using topological or geometric reasoning to help understand such qualitative features.

After Poincaré, the twentieth century saw this viewpoint expand and develop via pioneering work of Birkhoff (1930s), Kolmogorov (1950s), Smale, Arnol'd, and Moser (1960s), and others. The advent of computers and graphics has assisted experimental exploration, permitted approximate computation of solutions in many cases, and dramatized such phenomena as chaos. Nowadays dynamical systems has expanded far beyond its origins in celestial mechanics to illuminate many areas in physics, engineering, and chemistry, as well as biological and medical systems, population biology, economics, and so forth.

In the case of complex systems like the brain or the economy, the number of different relevant variables is very large. Moreover, firms may enter or leave a market, cells may grow or die; therefore the variables themselves are difficult to firmly specify. Yet the state of mathematical art dictates that any tractable mathematical model should not have too many variables, and that the variables it does have must be very clearly defined. As a result, conceptually understandable models are sure to be greatly simplified in comparison with the real systems. The goal is then to look for simplified models that are nevertheless useful. With this caveat firmly in mind, we now proceed to discuss some of the mathematics of dynamical systems theory.

In the following discussion, we assume only that the reader's background includes some calculus (so that the concept of derivative is familiar), and an acquaintance with matrices. Some references for further reading appear in section 2.4. (For a refresher on matrix algebra, see Hirsch and Smale, 1974.)

2.1 INTRODUCTORY CONCEPTS

In formulating the mathematical framework of dynamical systems, we may wish to consider time as progressing continuously (*continuous time*), or in evenly spaced discrete jumps (*discrete time*). This dichotomy corresponds to the differences between *differential equations* and *difference equations*; *flows* and *diffeomorphisms*. (These terms are defined below.)

We begin with the continuous time case, and proceed to discuss discrete time.

Differential Equations in Several Variables

In this section, we remind the reader of the basic terminology of differential equations. The real variable t will denote time (measured in unspecified units), and we use letters $x, y, z, \ldots$ to denote functions of time: $x = x(t)$, etc. These functions will be the (state) variables of the system under study. If we run out of letters, it is customary to use subscripts, as $x_1(t), x_2(t), \ldots, x_n(t)$ in the case of n variables, where n is some (possibly very large) positive integer. We denote by R^n the space of all n-tuples $(x_1, \ldots, x_n)$ of real numbers, representing n-dimensional Euclidean space.

The derivative (instantaneous rate of change) of x at time t is denoted $\dot{x}(t)$ (or sometimes $x'(t)$ or $(dx/dt)(t)$). [Note that $\dot{x}$ is the name of the function whose value at time t is $\dot{x}(t)$.]

The derivative $\dot{x}$ of x is a function that itself usually has a derivative, denoted $\ddot{x}$, the *second derivative* of x. This can continue indefinitely with the third derivative $\dddot{x}$, fourth derivative, etc. (though frequently only the first and second derivatives appear).

A *differential equation* in one variable (or one dimension) is simply an equation involving a function x and one or more of its derivatives. (Note that we are speaking exclusively of *ordinary* differential equations—equations in which all of the derivatives are with respect to a single variable (in this case time t). *Partial* differential equations involve partial derivatives of functions of more than one variable, and are not discussed in this chapter.)

Example 1 A simple frictionless mass-and-spring system is often modeled by the equation

$$m\ddot{x} + kx = 0.$$

Here x is a function of time representing the linear displacement of a mass, and m and k are constants, *mass* and the *spring constant* (or stiffness), respectively. To be clear, we emphasize that this means that for each time t, the number $m\ddot{x}(t) + kx(t)$ is zero. This is satisfied by sinusoidal oscillations in time.

Given this equation, the problem is to find a function $x(t)$ that satisfies this equation. Such a function is called a *solution* of the equation. In fact there will be very many such solutions, in this case one corresponding to each choice of the initial conditions $x(0)$ and $\dot{x}(0)$. The general solution (see Hirsch and Smale, 1974, or any beginning text on ordinary differential equations) is

$$x(t) = x(0)\cos((\sqrt{k/m})t) + (\sqrt{m/k})\dot{x}(0)\sin((\sqrt{k/m})t).$$

Typically a system has more than one state variable, in which case its evolution will be modeled by a *system* (or collection) of differential equations, as in

Example 2

$$\dot{x} = x + z$$

$$\dot{y} = 2x + y - z$$

$$\dot{z} = 3y + 4z.$$

Here one seeks three functions $x(t)$, $y(t)$, and $z(t)$, satisfying all three of these equations. This is a *linear* system of equations because it can be written as a single vector equation in the following matrix form: $\dot{X} = AX$, where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 0 & 3 & 4 \end{bmatrix},$$

and we are using ordinary matrix multiplication.

We say this is a *three-dimensional system* because it is expressed in terms of three state variables x, y, z (and hence a solution $(x(t), y(t), z(t))$ defines a curve in *three-dimensional* Euclidean space).

The virtue of a linear equation is that it is relatively easy to solve by standard methods. In a strong sense, linear equations are completely understood.

Of course, often systems of equations are not linear, like

Example 3

$$\ddot{x} = \dot{x} - y^3$$

$$\ddot{y} = -\dot{y} + x^3.$$

If a system is not linear, all bets are off: typically nonlinear equations cannot be solved explicitly. Nonlinear equations are important because (1) most systems are modeled by nonlinear, rather than linear, equations, and (2) solutions can have complicated and interesting behaviors, requiring various qualitative techniques of dynamical systems to analyze. (See examples 10 and 16.)

Vector Fields, Trajectories, and Flows

The matrix notation is very important because it provides us with a way of viewing any system as a *first-order system* involving perhaps more variables (first-order meaning that only the first derivatives of the variables are involved).

Example 4 The system in example 3 can be written as a first-order system by introducing new variables. Let $u = \dot{x}$, $v = \dot{y}$. Then our new state variables are x, u, y, v, and the system in example 3 is equivalent to the four-dimensional first-order system

$$\dot{x} = u$$

$$\dot{u} = u - y^3$$

$$\dot{y} = v$$

$$\dot{v} = -v + x^3.$$

From now on, we suppose that any given system of differential equations has already been put into this first-order vector form. In general, we then write it as

$$\dot{X} = F(X), \tag{1}$$

where $X = (x_1, \ldots, x_n)$ and F is a function from R^n to R^n. Note that for a *linear* system $\dot{X} = AX$, the function F is simply the *linear* function $F(X) = AX$. Within this framework, any differential equation can be specified simply by specifying the function F, called a *vector field*. In coordinates, we can express the value of F as

$$F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n)),$$

where the functions $F_1, F_2 \ldots, F_n$ are real-valued functions of n variables and are called the *component functions* of the vector field F. Thus, in example 4, we have

$$F(x, u, y, v) = (u, u - y^3, v, -v + x^3)$$

and the component functions are $F_1(x, u, y, v) = u$, $F_2(x, u, y, v) = u - y^3$, $F_3(x, u, y, v) = v$, $F_4(x, u, y, v) = -v + x^3$.

An *initial condition* for equation (1) is simply a choice of initial values

$$x_1(0), x_2(0), \ldots, x_n(0)$$

for each of the state variables. Equivalently, this is a choice of an initial vector $\mathbf{X}(0)$, which then determines a unique solution of equation (1).

Geometrically, you should think of a vector field on R^n as the assignment of a vector (direction and magnitude) at each point of R^n. Also, $\mathbf{X}(t)$ is to be interpreted, for each t, as the coordinates of a point in R^n, so that the function $\mathbf{X}$ represents a *trajectory*, or curve, through space. The point $\mathbf{X}(t)$ moves around continuously as t increases, tracing out its trajectory.

With this scheme, $\dot{\mathbf{X}}(t)$ represents the velocity vector of the trajectory of $\mathbf{X}$ at the point $\mathbf{X}(t)$, i.e., a vector tangent to the trajectory at that point, whose magnitude is given by the instantaneous speed of the point $\mathbf{X}(t)$ along the trajectory. Therefore equation (1) has a simple geometric interpretation: given the vector field F, solutions of equation (1) are simply trajectories that are everywhere tangent to F, and which have speed at each point equal to the magnitude of F. In terms of states, the system of equation (1) simply tells us how the rate of change $\dot{\mathbf{X}}$ of the state variable $\mathbf{X}$ at time t depends on its position $\mathbf{X}(t)$ at that time (figure 2.1).

We have now arrived at our new view of differential equations: by converting them into a system of equations in the form of equation (1), we think of the problem in the following geometric way: given a vector field $\boldsymbol{F}$, find the solution trajectories that pass through the field in the proper way.

Note that starting at two different points in space will produce two different solution trajectories (figure 2.2), unless the two points happen to lie on a single trajectory to begin with. Typically any given starting point determines a complete trajectory going forward and backward infinitely far in time. Moreover no two trajectories can cross. These are consequences of the so-called fundamental existence and uniqueness theorems for differential equations (see, e.g., Hirsch and Smale, 1974), and are true, for example, for any smooth and bounded vector field.

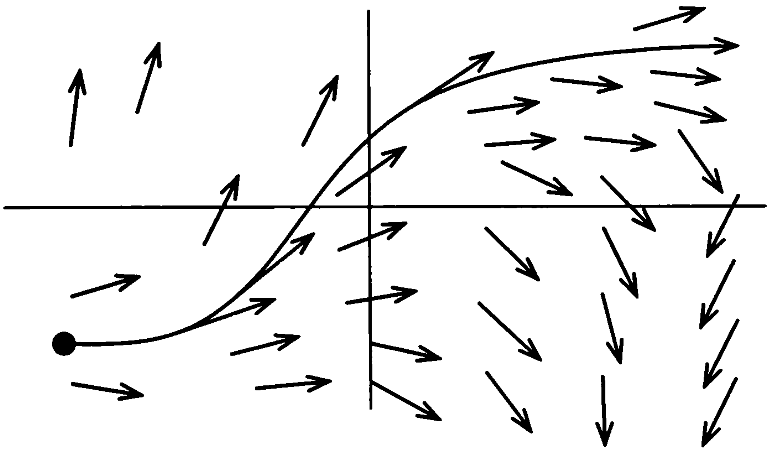

Figure 2.1 A vector field on R^2 along with a single solution trajectory.

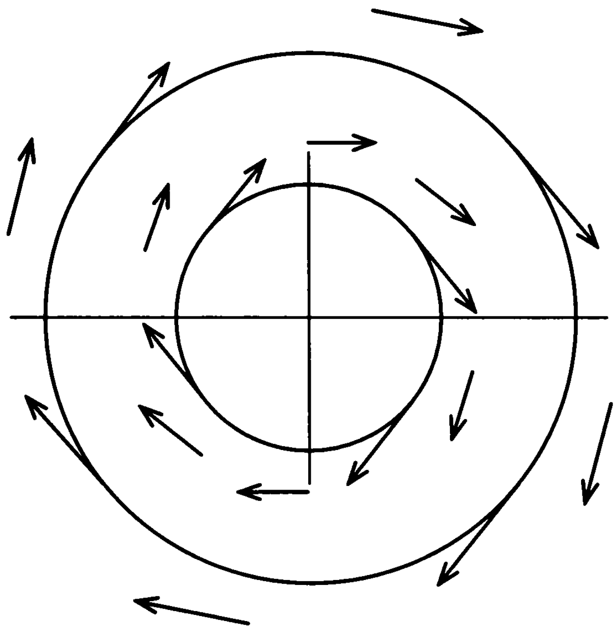

Figure 2.2 Two trajectories for the vector field $F(x, y) = (y, -x)$. These trajectories are periodic cycles.

The object of interest in dynamics, then, is the whole pattern of all the trajectories in the state space R^n. Each trajectory corresponds to a different solution of the equations (corresponding to different initial conditions). We now want to know various properties of this collection of trajectories and how to interpret them in terms of the behavior of the system being modeled. The picture of all the trajectories in the state space (also called the *phase space*) is called the *phase portrait*.

In dynamical systems one denotes the full solution of equation (1) by the *flow* $\phi(t, x)$. This is just a fancy notation for the position of a point x after it has followed its solution trajectory for a time t. For fixed x, $\phi(t, x)$, thought of as a function of t, is then simply a solution trajectory. For fixed t, $\phi(t, x)$,

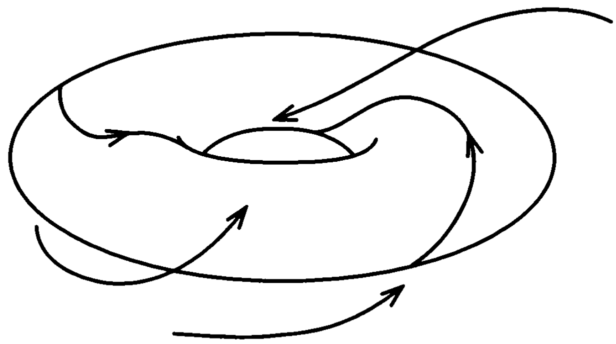

Figure 2.3 A toroidal manifold is shown along with the phase portraits of solutions on and approaching the manifold (in this case, a two-dimensional surface configured in three-dimensional space).

thought of as a function of x, is a transformation of the state space that moves each point along its own trajectory by the time t. The flow $\phi(t, x)$ of a vector field F, then, is in effect the complete solution of equation (1); it gives all of the solutions for all possible initial conditions.

Sometimes only a certain collection of solution trajectories is relevant, all of which happen to lie on some surface inside the full state space (or higher-dimensional analog of a surface, called a *manifold*). By restricting attention to such a manifold, one sometimes speaks of a vector field *defined on the manifold* (figure 2.3). (See Guillemin and Pollack, 1974, for a treatment of the subject of manifolds.)

Discrete Time Dynamics

Consider the simple differential equation in one variable

$$\dot{x} = g(x). \qquad (2)$$

The derivative $\dot{x} = dx/dt$ can be approximated by the difference quotient $\Delta x/\Delta t$, where $\Delta t = t_1 - t_0$ is a small difference between two time values, and $\Delta x = x(t_1) - x(t_0)$ is the corresponding difference in the values of the function x.

Hence equation (2) can be approximated by

$$\Delta x = g(x)\Delta t,$$

or, more explicitly,

$$x(t_1) - x(t_0) = g(x(t_0))(t_1 - t_0). \qquad (3)$$

Often we are interested in a discrete sequence of evenly spaced times, say $t = 0, 1, 2, 3, \ldots$. It is more common to use one of the letters i, j, k, l, m, n when denoting integers. With this change our equation becomes

$$x(k + 1) - x(k) = g(x(k)),$$

a so-called *difference equation*. We can simplify a little bit by writing $f(x) = g(x) + x$, so that equation (3) becomes

$$x(k+1) = f(x(k)) \qquad (k = 0, 1, 2, 3, \ldots) \tag{4}$$

for some function $f\colon R \to R$.

From equation (4), note that $x(k) = f(x(k-1))$, so that

$$x(k+1) = f(f(x(k-1)) \equiv f^2(x(k-1)),$$

where the notation $f^2(x)$ means $f(f(x))$, and in general $f^k(x)$ means $f(f(\ldots f(x)\ldots))$ (k times).

Continuing, we get

$$x(k+1) = f^{k+1}(x(0)) \qquad (k = 0, 1, 2, \ldots)$$

or, more simply,

$$x(k) = f^k(x(0)) \qquad (k = 1, 2, \ldots). \tag{5}$$

Equation (5) represents the most typical way of viewing a discrete dynamical system: it is one given by *iterating* a function f, starting from various initial values. Moreover x can be a real number or more commonly a point in R^n, in which case f is a function from R^n to R^n.

One should be careful to distinguish this function f from the vector field $\boldsymbol{F}$ described just previously, although the concepts are analogous since both describe a change that depends on the current state. One thinks of a vector field as a velocity vector at each point whose coordinates are the values of the coordinate functions of the vector field. In contrast, for a discrete dynamical system, the vector $f(x)$ is thought of as the new location of the point x after one iterate (unit of time).)

Iteration of the function f, starting with the initial value x_0, produces the (*forward*) *orbit* of x_0: the sequence

$$x_0, x_1, x_2, x_3, \ldots,$$

where $x_i = f^i(x_0)$ for $i = 0, 1, 2, \ldots$.

This is to be compared with the orbit of a vector field, which is a continuous trajectory or curve through the state space.

Exercise Let $f\colon R \to R$ be defined by $f(x) = 2x$. The reader is encouraged to investigate what happens to various points under iteration by f.

Time-One Maps, Poincaré Sections, and Diffeomorphisms

The finite difference approximation discussed above is only one way to arrive at a discrete dynamical system from a differential equation. More commonly, one considers the *time-one map of a flow*, or the *induced map on a Poincaré section*.

Given the flow $\phi(t, x)$ for a vector field F on R^n, one can define a function $f\colon R^n \to R^n$ by the rule

$$f(x) = \phi(1, x).$$

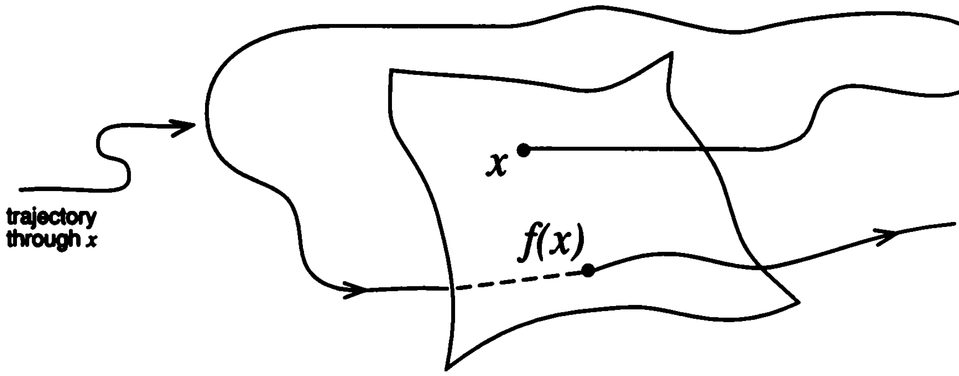

Figure 2.4 A Poincaré section for a vector field, showing a trajectory through x and its next intersection with the cross-section, at $f(x)$. A display of repeated iteration of the function can be very revealing about the dynamic behavior of the trajectories of the vector field.

This function is called the time-one map of the flow; its action is simply to move every point of the state space R^n along its solution trajectory by one unit of time. (Similarly we could just as well define the time-T map.)

Because of the standard properties of the flow of a nice-enough vector field, the time-one map will be a *diffeomorphism*, i.e. a differentiable mapping f of R^n that has a differentiable inverse (denoted f^{-1}). By means of f^{-1} one can move backward in time to obtain the *backward orbit* of x_0,

$$x_0, x_{-1}, x_{-2}, \ldots,$$

where $x_{-k} = (f^{-1})^k(x_0) \equiv f^{-k}(x_0)$. One can also speak of the *full orbit*

$$\ldots, x_{-2}, x_{-1}, x_0, x_1, \ldots.$$

Another very useful technique for passing from a flow to a diffeomorphism is to consider a Poincaré section, or cross section, of the vector field. This is a surface (of dimension one less than the dimension of the state space) that is nowhere parallel to the vector field. Starting at some point on this surface, if one follows the solution trajectory through that point, one will immediately leave the surface, travel around in state space, and then perhaps return to strike the surface once more (figure 2.4). Wherever this happens, one can define a *first-return mapping* which takes the initial point on the Poincaré section and sends it to the next intersection point of the trajectory with the section. (The trajectory may never again intersect the cross section, in which case the first-return map is not defined at that point.)

Orbits of the first-return map correspond closely with the trajectories of the flow, and one can often study the latter simply by investigating the former. The great merit of passing from a flow to a first-return map for some cross section is that one thereby reduces by one the number of dimensions of the problem, and this often makes a big difference in our ability to visualize the dynamics. In practice real three-dimensional systems of differential equations are often studied by taking a cross section and looking at the first-return map. Duffing's equation is interpreted this way in example 16 below.

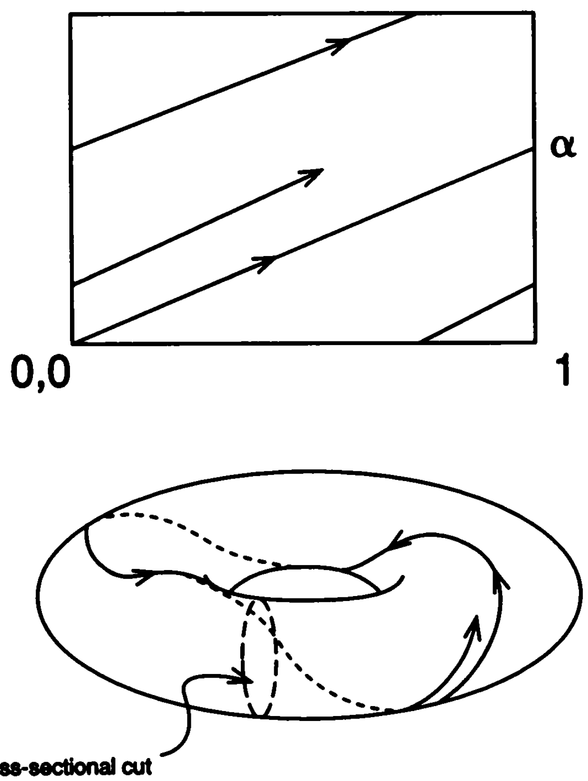

Figure 2.5 The top panel shows the torus opened up into a plane. Thus the left and right edges represent the same sectional cut through the torus, and the top and bottom edges similarly represent the same ring around the torus. A portion of an irrational trajectory is also shown. The bottom panel shows the usual view of the torus embedded in R^3.

Example 5 Consider the vector field on the torus given by irrational rotation, as follows. We represent the torus as the unit square in the plane with the opposite sides identified (glued together). On it define a constant vector field with the time-one map $F(x, y) = (1, \alpha)$, where α is some irrational number between 0 and 1. The solution trajectories are then lines parallel to the vector $(1, \alpha)$ through every point on the torus (figure 2.5). (When a trajectory reaches the right-hand edge, it reappears at the corresponding point on the left, and similarly for top and bottom.)

We can take as our Poincaré section the vertical circle indicated by the dotted line. The reader should convince himself or herself that the first return map in this case is the irrational rotation of the circle $f(x) = x + \alpha \pmod 1$. See example 13 below for this notation.

Endomorphisms

Not all diffeomorphisms arise as time-one mappings of flows. Given a random diffeomorphism, one can still iterate it and thereby produce a dynamical system. In fact it is not necessary that the function be invertible: any function $f: R^n \to R^n$ can be iterated and so the forward orbits are always defined (but perhaps not the backward orbits). A possibly noninvertible function from a space to itself is called an *endomorphism* (as opposed, e.g., to a function from one space to another). Any such thing can be iterated and thought of as a dynamical system, and in fact some very simple examples can produce very interesting dynamics.

Example 6 The subject of much current research is the study of the dynamics of functions on the unit interval [0, 1] like $f_a(x) = ax(1 - x)$, where a is some positive parameter, each different value of which yields a different function. The problem here is to understand the dynamics of the functions f_a for different values of a, and how the dynamics changes or *bifurcates* as the parameter (hence the function) is changed.

For values of a between 0 and 1, the reader can discover that every point in [0, 1] simply tends to 0 under iteration. As a increases past 1, 0 becomes a repelling fixed point and a new attracting fixed point appears. The system has undergone a *bifurcation*. Further increase in the value of a leads to successive bifurcations in which the attracting fixed point splits into two attracting points of period two, each of which later splits into pairs of period-four points, etc. At the end of this so-called period doubling cascade, the map becomes chaotic. The interested reader should see Devaney (1986) for more on this topic.

Attractors and Bifurcations

There is no general agreement on the precise definition of an attractor, but the basic idea is straightforward. Here is one version.

Let F be a vector field on $\mathbf{R}^n$, with flow ϕ. A closed set $A \subset \mathbf{R}^n$ is an *attractor* for this flow if (1) all initial conditions sufficiently close to A have trajectories that tend to A as time progresses, (2) all trajectories that start in A remain there, and (3) A contains no smaller closed subsets with properties (1) and (2).

More precisely, let $d(x, A)$ denote the distance between a point x and the set A. Condition (1) means there exists $\varepsilon > 0$ such that $d(x, A) < \varepsilon$ implies $d(\phi_t(x), A) \to 0$ as $t \to +\infty$.

Condition (3) follows if A contains a dense orbit, that is, a trajectory that visits every region of A infinitely often. Sometimes this stronger condition is used instead of (3).

A similar definition can be made for diffeomorphisms (i.e., for the discrete case).

Attractors are important because they represent the long-term states of systems. If we imagine that most real systems have already been evolving for some time before we observe them, then we would expect that attractors represent the behaviors we actually observe in nature, at least for systems that have settled into their long-term behaviors.

Often, as in a marble rolling around in a bowl, the attractor is simply the fixed point corresponding to the resting position at the bottom [attracting fixed point, or sink: examples 8, 11(ii)]. Other times the attractor is a periodic orbit representing a steady-state oscillating behavior [attracting periodic orbit: example 9 ($a < 0$)].

One of the insights afforded us by dynamical systems theory is that these are not the only regular long-term behaviors for systems: there are also *strange* or *chaotic attractors*. In this case the attractor contains within it expanding directions that force nearby trajectories to rapidly diverge from one another as time progresses (examples 15, 16). Often such attractors have a fractal geometric structure, with irregularity repeated at arbitrarily small scales.

A point of fundamental interest is to understand how an attractor changes as the dynamical system (vector field, differential equation, diffeomorphism) itself is changed. The system may contain various parameters that can take on different values and lead to different dynamical behaviors. As the parameters change gradually, it is of great importance to know how the attractors change.

Often, a small change in the parameters will lead to a correspondingly small change in the shape of the attractor, but no change in its qualitative features. Other times, a parameter value is reached at which a sudden change in the qualitative type of the attractor occurs. When this happens, we say the system has undergone a *bifurcation*.

The study of bifurcations is a large subject, but we can say a few words here about the simplest cases of bifurcation of an attracting fixed point (see Guckenheimer and Holmes, 1983, for more information).

Consider the following equations:

$$\dot{x} = a - x^2 \qquad \text{(saddle-node)}, \tag{i}$$

$$\dot{x} = y + x(a - x^2 - y^2) \tag{ii}$$

$$\dot{y} = -x + y(a - x^2 - y^2) \qquad \text{(Hopf)}.$$

Next to the name of each of two standard bifurcations is an equation or set of equations that exhibit that bifurcation as the parameter a passes through the value zero. The following sequence of diagrams illustrates what happens as a system undergoes a saddle-node bifurcation in one dimension (figure 2.6).

The case of the Hopf bifurcation is described further in example 9 below.

One point of importance here is that a "generic" (i.e., typical) system of equations

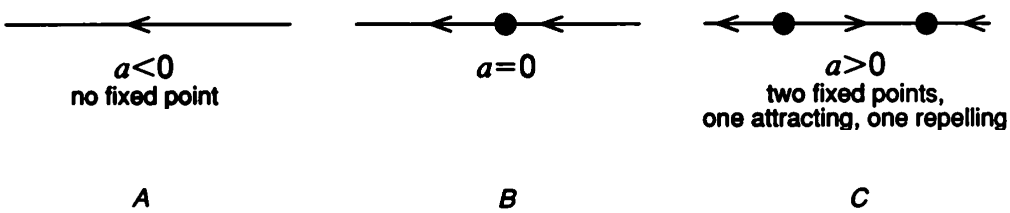

Figure 2.6 With respect to equation (i) above, with $a < 0$ there is no fixed point, as shown in A; with $a = 0$ there is a saddle as in B; and with $a > 0$, there are two fixed points, one stable and one unstable, as shown in C.

$$\dot{X} = F_\mu(X),$$

depending on one real parameter μ, will, at a bifurcation value for a fixed point, undergo one of these bifurcations (Guckenheimer and Holmes, 1983). That is, along some one- or two-dimensional subset of the state space containing the equilibrium, the qualitative behavior of the system will look like one of these. Therefore understanding the bifurcations (i) and (ii) means understanding all at once the way attracting fixed points bifurcate for generic systems of even very large dimension.

There are other standard types of bifurcations that can occur when further constraints are imposed on the systems being considered. Any specific family of systems might have a nonstandard bifurcation, but then a small pertubation of the family may produce a standard one. Furthermore, systems that depend on two or more parameters will generally undergo more complicated types of bifurcations (as studied in the field of bifurcation theory).

2.2 STABILITY AND CHAOS

To introduce some further concepts, including *stability* and *chaos*, we devote the remainder of this chapter to a series of basic illustrative examples.

Example 7 Here we return to consider the frictionless mass-and-spring system of example 1 (figure 2.7),

$$m\ddot{x} + kx = 0.$$

For simplicity we take m = k = 1. Letting $\dot{x} = u$, we obtain the two-dimensional first-order system

$$\dot{x} = u$$

$$\dot{u} = -x.$$

Here, the vector field is simply $F(x, u) = (u, -x)$, and the solution is

$$x(t) = x_0 \cos(t) + u_0 \sin(t)$$

for the initial conditions $x(0) = x_0$, $u(0) = u_0$ (see Hirsch and Smale, 1974, to learn how to solve such equations). The phase portrait in the phase plane (i.e.,

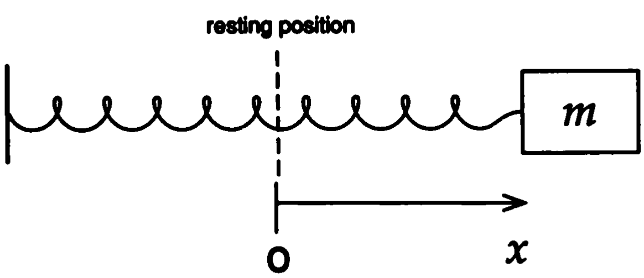

Figure 2.7 A simple, frictionless oscillator with a mass (m) at position x and with resting position 0.

state space R^2) then consists of concentric circles centered at the origin (figure 2.8).

An initial condition corresponds to some starting point in the plane, and then the state (x, u) of the system evolves according to $(x_0 \cos(t) + u_0 \sin(t), -x_0 \sin(t) + u_0 \cos(t))$: i.e., it follows the circular trajectories in a clockwise direction. (It is easier to see this in the case $u_0 = 0$, when the solution is of the form $(x_0 \cos(t), -x_0 \sin(t))$.)

For the physical system, this corresponds to the mass oscillating back and forth periodically about its equilibrium position at the origin. Since x represents position (distance from rest position) and u velocity, we see that the speed of the mass is greatest as it is passing through its equilibrium position, and the speed is zero when the spring is stretched or compressed the most.

The origin (0, 0) corresponds to the state in which the mass is sitting at rest at its equilibrium position. This is called a *fixed point* of the flow, or a *zero* of the vector field. A system starting out at a fixed point will remain there forever.

An important question about a fixed point is: Is it stable? There are two notions of stability of fixed points, as follows. A fixed point is *Lyapunov-stable* if points near the fixed point continue to remain nearby forever. The fixed point is *asymptotically* stable if nearby points actually tend toward the fixed point as time progresses.

For our mass-and-spring example, the origin is Lyapunov-stable but not asymptotically stable: points near the origin follow circular trajectories that remain nearby but do not tend to the origin in the limit as $t \to \infty$.

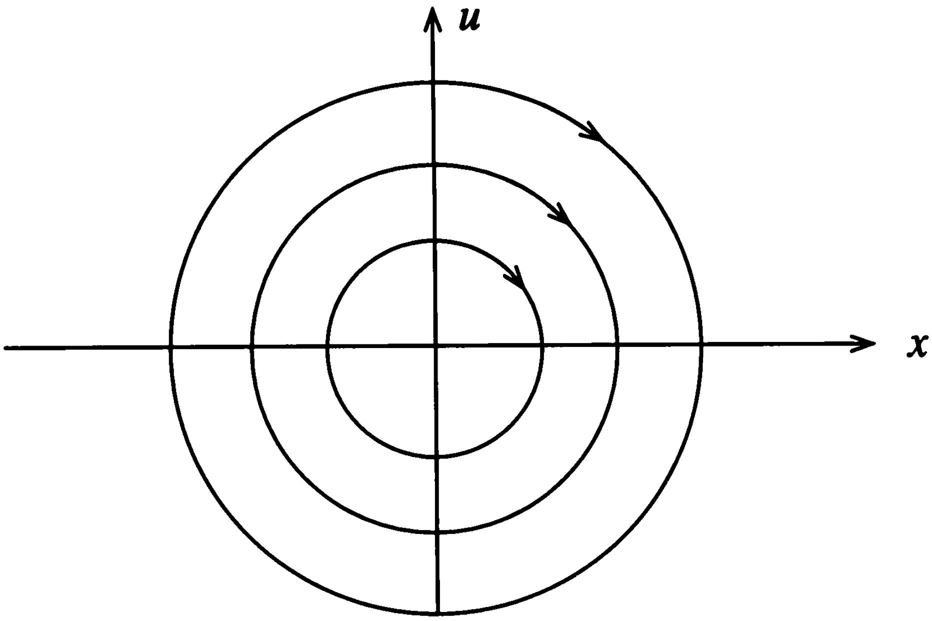

Figure 2.8 With no friction, the oscillator will sweep out concentric circles in the phase plane. The diameter of the circle depends on the initial state. Larger circles mean wider excursions along x as well as larger peak velocities along $\dot{x}$.

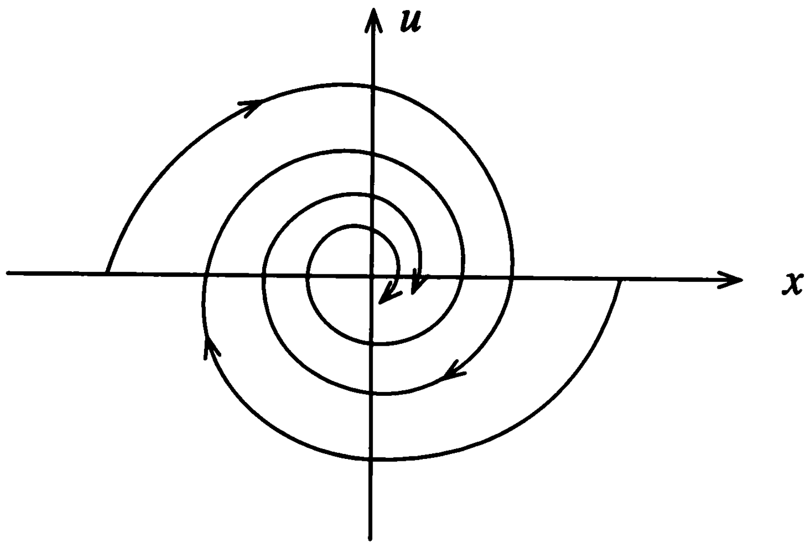

Figure 2.9 The phase portrait of a mass-and-spring system with a friction term. Amplitude of x and $\dot{x}$ approach zero over time.

Example 8 Since the drag force of sliding friction is roughly proportional to velocity, adding friction to the mass-and-spring system of the previous example is typically modeled by adding a first-derivative term to the equation, as in

$$\ddot{x} + \dot{x} + x = 0.$$

In terms of our variables x, u,

$$\dot{x} = u$$

$$\dot{u} = -x - u.$$

The solution trajectories of this system turn out to spiral down toward the origin (figure 2.9). Since the vector field is $F(x, u) = (u, -x - u)$, $(0, 0)$ is again a fixed point. By inspecting the phase portrait, it is clear that this fixed point is both Lyapunov-stable and asymptotically stable. We also say it is an *attracting fixed point,* because all nearby points tend in toward the origin as time proceeds. (An attracting fixed point is the simplest version of an *attractor.*) This makes physical sense because we expect friction to cause the oscillations to die down toward the resting position.

Example 9 Let a be a real parameter, and consider the system

$$\dot{x} = y + (a - x^2 - y^2)x$$

$$\dot{y} = -x + (a - x^2 - y^2)y.$$

For $a < 0$ we have a single attracting fixed point at the origin. Observe what happens when a is gradually increased. When it reaches $a = 0$, we still barely have a single attracting fixed point toward which every trajectory tends. When $a > 0$, the origin becomes a *repelling* fixed point—i.e., a fixed

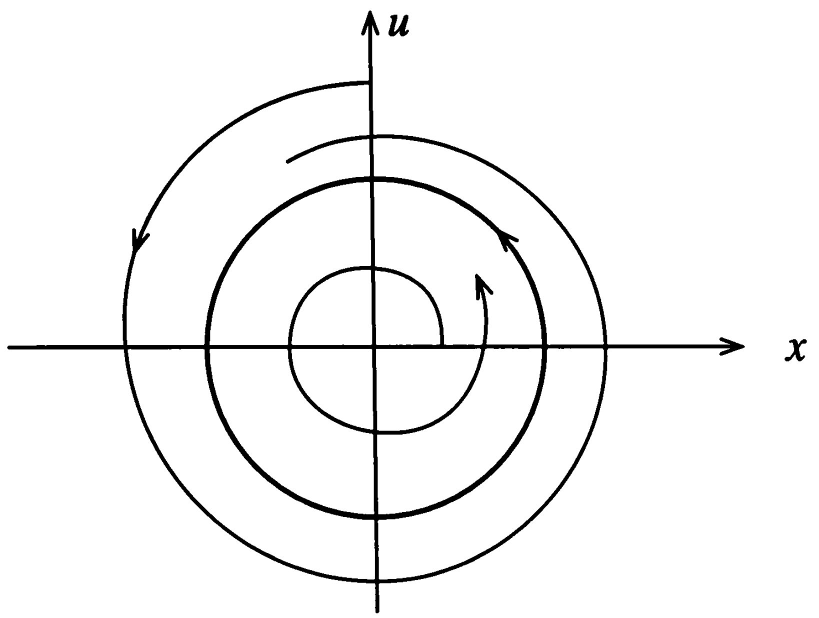

Figure 2.10 Phase portrait for $a > 0$. Note the attracting periodic orbit shown with the thicker line. The origin is a repelling fixed point.

point toward which trajectories tend as time runs backward to $-\infty$. Springing out from the origin is a new attractor: an *attracting cycle* (or attracting periodic orbit). All trajectories, except for the fixed point at the origin, tend toward this new cycle. This phemomenon is a simple example of a *Hopf bifurcation,* in which, as a parameter changes across some critical value (in this case 0), an attracting fixed point gives birth to an attracting cycle and itself becomes repelling (figure 2.10).

Example 10 A simple frictionless pendulum, as shown in figure 2.11, can be described by the variables x and u, where x is the angle of deviation from the vertical, and $u = \dot{x}$ is the angular velocity (figure 2.11).

Newton's laws lead to the equations of motion

$$\dot{x} = u$$
$$\dot{y} = -c \sin x,$$

where c is a constant proportional to the length of the pendulum.

Fixed points appear whenever the vector $(u, -c \sin x)$ is zero; i.e., $u = 0$ and $x = k\pi$, and k is any integer. The phase portrait is shown in figure 2.12.

Here the origin is Lyapunov-stable, while the point $p = (\pi, 0)$ is an unstable equilibrium called a *saddle point.* Two trajectories tend asymptotically toward p in forward time, two in backward time, and other trajectories come near but then veer away. This point corresponds to the pendulum at rest pointed straight up, delicately balanced. Any small perturbation will tend to push it onto one of the nearby trajectories—either one cycling around a rest point (oscillating behavior), or one moving off toward infinity to the right or left (the pendulum rotates continuously around in one direction).

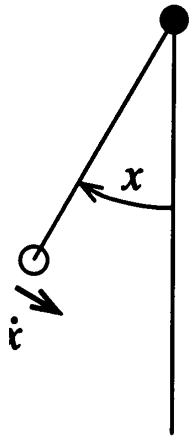

Figure 2.11 A simple pendulum has variables x, position, and $\dot{x}$, velocity. Part of its phase portrait, without friction, resembles figure 2.8, but is expanded in figure 2.12.

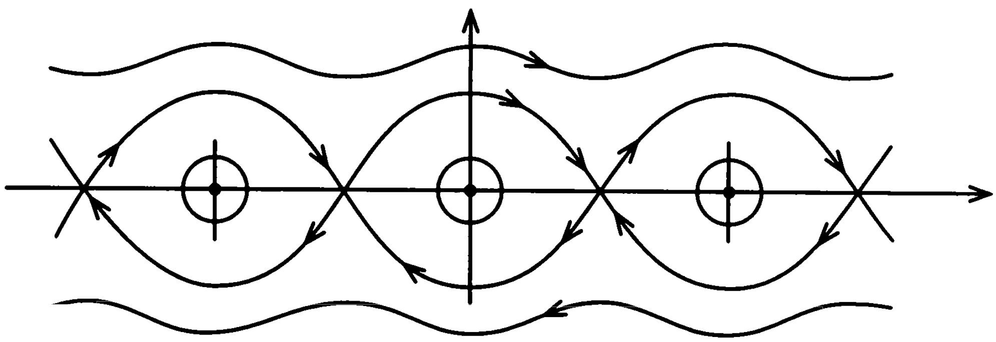

Figure 2.12 The phase portrait of a frictionless pendulum. The three dots represent the same critical point (at the bottom). The sinusoids are the separatrices leading to the unstable saddle points when the pendulum remains straight up. The wavy lines at top and bottom are trajectories that spin around over the top of the pendulum in opposite directions.

Example 11 We now turn to similar considerations for some discrete-time dynamical systems, beginning with three simple functions of one variable:

(i) $f(x) = 2x$. This simple diffeomorphism has a single fixed point (zero) which is *repelling*, meaning that nearby points move away under iteration (or: nearby points tend to zero under backward iteration, i.e. application of the inverse f^{-1}). All other points tend to infinity under iteration. Figure 2.13A shows behavior beginning at $\pm 1/2$.

(ii) $g(x) = f^{-1}(x) = (1/2)x$. Here the origin is an attracting fixed point, and all points on R tend to 0 under iteration.

(iii) $h(x) = e^x - 1$. Here, points to the left of the origin tend, under interation, to 0, while points to the right tend away. Since 0 is then neither an attracting nor a repelling fixed point, it is called a *neutral* fixed point.

Note that $|f'(0)| > 1$, $|g'(0)| < 1$, and $|h'(0)| = 1$. This is no accident: If x_0 is a fixed point for f, and $|f'(x_0)| > 1$, then x_0 is repelling; if less than 1, x_0 is attracting; if equal to 1, the fixed point may be either repelling, attracting, or neutral.

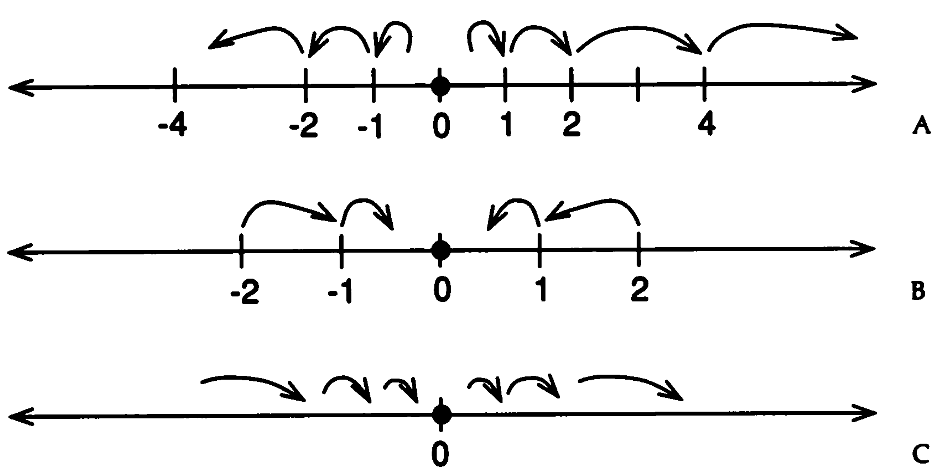

Figure 2.13 *A, B,* and *C* illustrate iteration of the functions $f(x) = 2x$, $g(x) = (1/2)x$, and $h(x) = e^x - 1$, respectively.

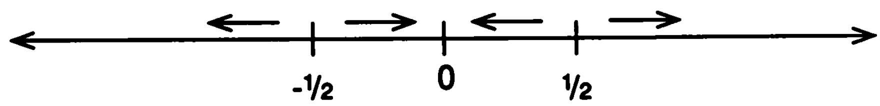

Figure 2.14 Illustration of example 12. Beginning near $-1/2$ or $+1/2$, points either converge toward 0 or tend to infinity.

Exercise Find an example of a function $f: R \to R$ such that 0 is a repelling fixed point, but $|f'(0)| = 1$.

Example 12 Let $f(x) = x^3 - (3/4)x$ (figure 2.14). This function has three fixed points: $x = 0, 1/2, -1/2$. The reader should check that 0 is attracting and $1/2, -1/2$ are repelling.

Example 13 The circle can be represented by the unit interval $[0, 1]$ with the endpoints identified; since 0 and 1 are then two representatives for the same point, we omit 1 and take the notation $[0, 1)$ as representing all magnitudes from 0 to 1, but excluding 1 itself.

We think of the numbers in $[0, 1)$, then, as representing angles on the circle (where a full turn is taken to have angle 1 instead of 2π to simplify notation). Addition in $[0, 1)$ is then just the addition of angles on the circle: $(1/2) + (3/4) \pmod 1 = 5/4 \pmod 1 = 1/4 \pmod 1$. Here "mod 1" tells us to add or subtract an integer so that the result lies in $[0, 1)$.

Consider the rotation $R_a(x) = x + a \pmod 1$ for various rotation angles a.

(i) $a = 1/3$. Here R_a has no fixed points, but every point of the circle is a *periodic point* of period 3: after three iterations of R_a, each point return to itself (figure 2.15).

In general, we say a point x_0 is a *periodic point of period k for a map f* if $f^k(x_0) = x_0$, and if k is the least positive integer with this property. A periodic

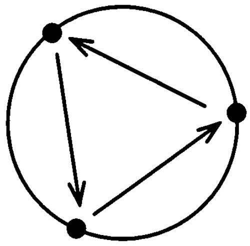

Figure 2.15 A graphic representation of a periodic point with period 3. Successive rotations of multiples of 1/3 lead only between the three angles shown.

point of period 1 is a fixed point. An attracting periodic point for f of period k is one which is an attracting *fixed* point for the iterate f^k. Similar definitions hold for repelling and neutral periodic points. In the case of the rotation by 1/3, every point is a neutral periodic point of period 3.

(ii) $a = 2/5$. Check that each point of the circle $[0, 1)$ is a periodic point of period 5.

(iii) $a = 1/\sqrt{2}$. In this case R_a has no periodic points of any period, because the angle of rotation is irrational. Instead of repeating after a finite number of steps, the forward orbit of each point fills in the whole circle more and more densely. We say that the map is *transitive,* because it has a dense orbit (and in fact every orbit is dense).

Example 14 Let $f: [0, 1) \rightarrow [0, 1)$ be the *angle doubling map of the circle,* defined by $f(x) = 2x \pmod 1$. This map exhibits the basic features of "chaos," namely, *sensitive dependence on initial conditions, transitivity,* and *dense periodic points.*

Sensitive dependence on initial conditions means that any two nearby starting points rapidly diverge from each other as iteration continues. That is, if x and y are two nearby points on the circle, then the distance between $f^k(x)$ and $f^k(y)$ grows (exponentially fast) with k. The reason is that one can see from the definition of f that the distance between any two nearby points simply doubles with every iterate—until the distance between them is more than 1/4. See below for more on this concept.

Dense periodic points: Any small interval on the circle contains a periodic point (of some period) for f. To see this, the reader can verify that if $x = p/(2^k - 1)$ for any integer p, that $f^k(x) = x \pmod 1$.

In particular, there are infinitely many periodic points for f (though only finitely many for any given period).

Transitivity: This simply means there is a dense orbit. This is not hard to prove, but we will not do so here. (See Devaney, 1986.)

Example 15 The *solenoid map* is a three-dimensional diffeomorphism defined on the "solid torus" $S = S^1 \times D^2$, where S^1 denotes the unit circle and $D^2 = \{(x, y) \in R^2: x^2 + y^2 \leqslant 1\}$ is the unit disk in the plane (figure 2.16).

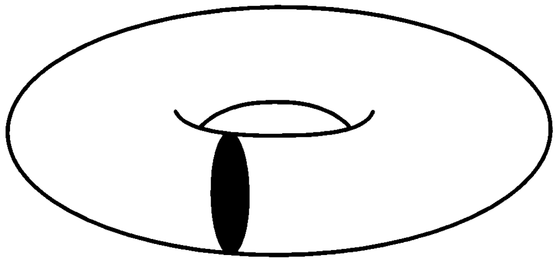

Figure 2.16 This solenoid map is defined on a solid torus, as opposed to the two-dimensional manifold of figure 2.5.

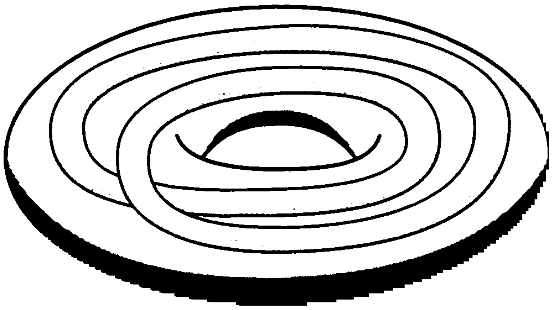

Figure 2.17 A single iteration of the mapping in example 15 generates a longer, narrower solid torus that wraps around twice inside the first one.

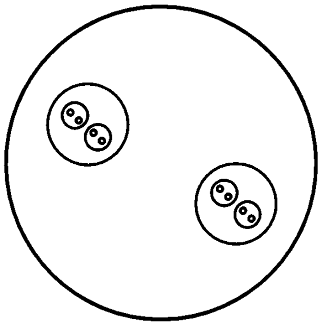

Figure 2.18 Repeated iteration of the mapping in example 15 embeds additional tori within each previous torus. This display is a cross-sectional cut through three iterations.

The mapping $f: S \to S$ is defined by

$$f(\theta, x, y) = (2\theta, (1/4)x + (1/2)\cos\theta, (1/4)y + (1/2)\sin\theta).$$

The action of f is to stretch the solid torus out, wrap it around twice, and place it inside the original solid torus (figure 2.17).

Then f^2 takes the original solid torus to a very thin one wrapped four times around the original, etc. The resulting attractor, obtained as the intersection of all these thinner and thinner tubes, is called a *solenoid* (figure 2.18).

Every point of the original solid torus tends toward this solenoid; points on the solenoid itself experience a stretching apart very similar to that of the angle-doubling map of the circle. In fact, f restricted to the solenoid exhibits all the chaotic properties of the angle-doubling map, so it is called a *chaotic attractor*. (See Devaney, 1986.)

Example 16 *Duffing's equation.* We can illustrate a few more ideas with the equation

$$\ddot{x} + \delta\dot{x} - x + x^3 = \gamma\cos(\omega t),$$

used to model the forced vibration of a stiff metal beam suspended vertically between two fixed magnets on either side (see Guckenheimer and Holmes, 1983; Vector Fields, Trajectories, and Flows, above). Here δ is a small positive constant, γ represents the magnitude of the periodic forcing term, and ω represents the frequency of forcing. Though there is no general solution expressible in terms of elementary formulas, we can still study the system as follows.

Writing this as a first-order system, we get (changing x to $\dot{u}$):

$$\dot{u} = v$$

$$\dot{v} = u - u^3 - \delta v + \gamma\cos(\omega t).$$

Note that the vector field is time-dependent, so this is a *nonautonomous* system. To deal with this, we convert the time variable into a third-space variable as follows:

$$\dot{u} = u$$

$$\dot{v} = u - u^3 - \delta v + \gamma\cos(\omega\theta).$$

$$\dot{\theta} = 1.$$

Here u and v are as before and θ is a new angular variable which increases at constant rate from zero to $2\pi/\omega$ and then repeats. (This is permitted because of the periodicity of cosine.) Therefore we can think of θ as moving around a circle of length $2\pi/\omega$.

The state space is then $R^2 \times S^1$: the space of all triples (u, v, θ), where u and v are real numbers and θ represents an angle. In this case a convenient cross-section is the two-dimensional set $\Sigma = \{(u, v, \theta)\colon \theta = 0\}$, where we can take u and v to be coordinates for Σ.

The first-return map $f\colon \Sigma \to \Sigma$ of the flow to this cross-section is then simply the time-$2\pi/\omega$ map of the flow for the three-dimensional system above, restricted to Σ. This is easy to present graphically by plotting the orbits of various points with a computer: start at any point $(u, v, 0)$, and plot the u and v coordinates of the trajectory after times in multiples of $2\pi/\omega$. Here is the picture one obtains for the values $\omega = 1$, $\delta = 0.2$, $\gamma = 0.3$ (figure 2.19).

The result is apparently a chaotic attractor (viewing it only in cross-section). That is, nearby initial conditions tend closer to this set as time progresses, but within the set, nearby orbits diverge from one another. See Guckenheimer and Holmes (1983) for a more complete discussion of this situation.

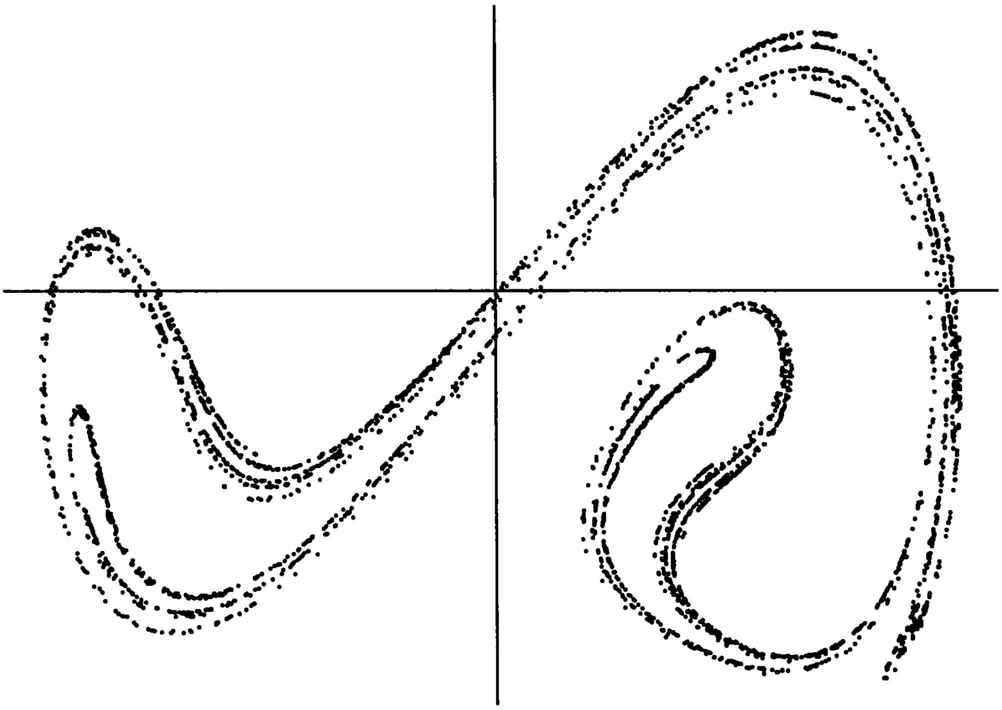

Figure 2.19 The Poincaré section (first-return map) of Duffing's equation is shown after 1000 iterations. This chaotic attractor shows how points will converge to this complex shape under iteration, yet within this general pattern nearby points diverge from one another.

2.3 CONCLUSIONS

We have made a general survey of what dynamical systems are and how they are analyzed by mathematicians. It should be clear that one way this research has progressed is by relaxing the search for specific solutions for specific initial conditions. The sensitive properties of dynamical systems force us to do so, since very small differences in initial conditions may be magnified over a short time to dramatically different states.

Instead, a wide range of methods have been developed during the 20th century for describing and evaluating the qualitative properties of dynamic models. These qualitative and topological properties turn out to offer many insights into the behavior of actual complex systems.

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Guide to Further Reading

This has been a very elementary introduction to some basic ideas of mathematical dynamical systems. For the reader wishing to learn more, the visual introduction to dynamics by Abraham and Shaw (1982) is especially accessible. For a general mathematical background, read Hirsch (1984). This article begins with an excellent historical discussion of dynamical systems, requiring no technical background, followed by some general discussion at a higher technical level than this chapter. The last section of the article discusses so-called monotone flows, important in population dynamics. Also excellent reading is the little paperback by Steven Smale (1980), a collection of articles and essays by one of the founders of the modern era of dynamical systems. It contains a reprint of his important 1967 mathematical survey article "Differentiable Dynamical Systems," along with nontechnical discussions, reminiscences, and essays on economics and catastrophe theory. For a basic textbook on dynamics, try Devaney (1986), which has been an influential introduction to discrete dynamical systems. Much of the book can be read with only a background in calculus, and it provides a valuable overview of many of the key ideas of modern dynamical systems. Hirsch and Smale jointly authored a popular text (1974) which gives a thorough introduction to the basic concepts, with many examples and applications. Only an exposure to the calculus of several variables is required; the reader need have no prior familiarity with linear algebra or differential equations. For a text with a more applied flavor, there is Guckenheimer and Holmes (1983), which concentrates on continuous time dynamics. Many standard examples of nonlinear differential equations are discussed and analyzed, and there is some treatment of bifurcation theory. This is an excellent general reference for those who have already been exposed to a first course in differential equations, such as Hirsch and Smale. Also excellent is the result and detailed survey by Robinson (1995). For those wishing to learn more about manifolds, look to Guillemin and Pollack (1974) as a standard and accessible mathematics text. Prerequisites include linear algebra and a year of analysis past calculus. An excellent and more advanced treatment is Hirsch (1976).

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